CHARACTERISTIC NUMERICAL RELATIVITY APPLIED TO HYDRODYNAMIC STUDIES OF NEUTRON STARS

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We present tests and results of a new axisymmetric, fully general relativistic code capable of solving the coupled Einstein-matter system for a perfect fluid matter field. Our implementation is based on the Bondi metric, by which the spacetime is foliated with a family of outgoing light cones. We use high-resolution shock-capturing schemes to solve the fluid equations. The code can accurately maintain long-term stability of a spherically symmetric, relativistic, polytropic equilibrium model of a neutron star. In axisymmetry, we demonstrate global energy conservation of a perturbed neutron star in a compactified spacetime, for which the total energy radiated away by gravitational waves corresponds to a significant fraction of the Bondi mass.

1 Characteristic numerical relativity and hydrodynamics

1.1 Introduction

We describe a new axisymmetric, fully general relativistic numerical code evolving the Einstein equations along with perfect fluid matter and apply it to studies of neutron stars. Our numerical implementation of the field equations of general relativity is based on the light cone formalism of Bondi[2] and Tamburino-Winicour[14]. The light cone approach has a number of advantages compared to spacelike foliations. i) It is physically motivated; the light cones offer a simple and unambiguous physical gauge on which to base the numerical spacetime grid. ii) It is unconstrained; the evolved variables capture rather directly the true degrees of freedom of the gravitational field. iii) It is very efficient; there are but two partial differential equations to solve, along with a set of radial integrations. iv) It allows for well defined compactification of the domain, which leads to perfect outer boundary conditions. v) Finally, and may be most importantly, the above theoretical advantages have been shown in a series of works to translate to remarkably numerically robust and stable codes, see for example [7]. For a recent review of the approach see Winicour[15].

The broad target of this project is the detailed investigation of neutron star dynamics using numerical relativity. A prerequisite for such studies is the development of very accurate and long-term stable general relativistic codes. It would seem that the feature list of the characteristic approach makes it ideal for such

studies. One serious bottleneck of the approach immediately recognized by the trained relativist is the breakdown of a lightlike coordinate system in the emergence of light caustics. Interestingly though, due to its quasispherical nature, no matter how agitated a neutron star, it is unlikely to focus the light cones emanating from its interior. A limitation of more technical nature is the existence of a somewhat restrictive Courant-Friedrichs-Lewy-condition for explicit algorithms in multidimensions. Implicit evolution schemes would help in this direction.

The incorporation of fluid matter fields in the characteristic formulation of the Einstein equation was considered as early as 1983[8] and numerical results were reported in reference[1]. A different line of attack was initiated recently[11] which brings into the considerations the modern machinery from Computational Fluid Mechanics. In this approach, the evolution equations for the matter fields are solved using relativistic high-resolution shock-capturing schemes[4,11] (HRSC schemes) based upon (exact or approximate) Riemann solvers. At this early stage of our investigations, the ability of capturing shock waves in the hydrodynamics is not essential. Nevertheless it will be so when studying core-collapse scenarios. The implementation of HRSC schemes in a general characteristic code without imposing symmetry conditions is the current subject of a collaboration (GRACE, General relativistic astrophysics via characteristic evolution).

In this work we restrict ourselves to axisymmetric spacetimes. The code has been developed building on previous work by Gómez, Papadopoulos and Winicour[6] (GPW), which constructed an axisymmetric characteristic vacuum code, to which we have added a perfect fluid matter field. Applications in spherical, dynamic black hole spacetimes have been presented in references[11,12]. Studies of the spherically symmetric collapse of supermassive stars are discussed in reference[9].

1.2 Fundamental equations

The main equations we have to solve for the problem we are aiming at are the Einstein equation

$$G_{ab} - \kappa T_{ab} = 0 \tag{1}$$

and the conservation equation of the stress-energy tensor T_{ab},

$$\nabla^a T_{ab} = 0. \tag{2}$$

The stress-energy tensor is chosen as that of a perfect fluid,

$$T_{ab} = \rho h u_a u_b + p g_{ab}, \tag{3}$$

where ρ denotes the mass density, h the relativistic specific enthalpy and u_a the four-velocity of the fluid. The pressure p is subject to an equation of state. We choose a perfect fluid equation of state

$$p = (\Gamma - 1)\rho\epsilon, \tag{4}$$

where ϵ denotes the specific internal energy of the fluid, related to the specific enthalpy as $h = 1 + \epsilon + \frac{p}{\rho}$. In addition we assume conservation of baryonic matter,

$$\nabla^a (\rho u_a) = 0. \tag{5}$$

We solve the above equations using the standard coordinate system of characteristic numerical relativity (see Figure 1).

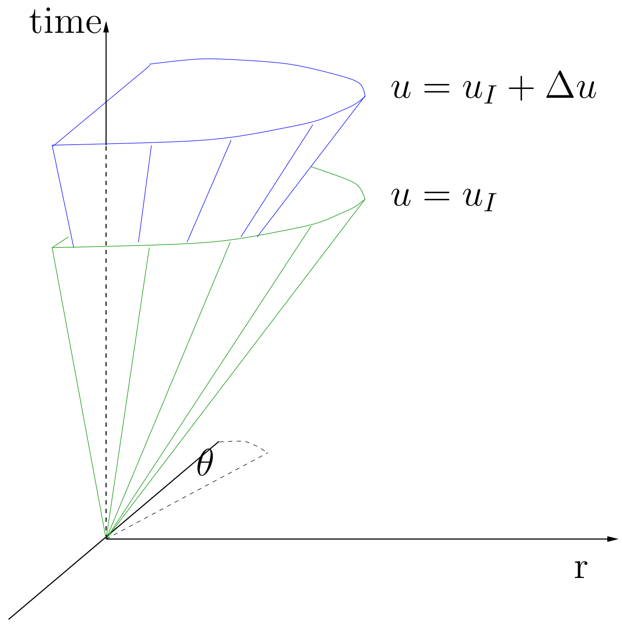

Figure 1. Foliation of the spacetime by null hypersurfaces (light cones): Starting from the geodesic of a freely falling particle, which defines the origin of the coordinate system, surface forming light rays are emitted. We define a new time coordinate u to be constant along each outgoing light cone. The Killing coordinate ϕ is suppressed in the diagram. The origin of the coordinate system will lie (in the axisymmetric case) on the symmetry axis of the star. If the system has additional reflection symmetry with respect to the equator, the origin is actually at the center of the star.

More precisely, we use the Bondi metric

$$\begin{aligned} ds^2 = &-(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}du\ dr - 2Ur^2e^{2\gamma}du\ d\theta \\ &+r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}sin^2\theta\ d\phi^2) \end{aligned} \tag{6}$$

with null coordinate u, radial coordinate r, azimuthal coordinate θ and the spherical coordinate ϕ, which is a Killing coordinate. The four metric fields β, γ, U and V are determined by solving the Einstein equation (1). With a choice of a coordinate chart (x^0, x^i), the hydrodynamic equations (2) and (5) are written as an explicit hyperbolic system, well-suited for numerical applications, as detailed in references[11,13].

1.3 Numerical implementation

The numerical implementation of the Einstein equation (1) closely follows that of GPW. Using the Bondi metric (6), equation (1) splits into hypersurface equations on

each light cone (for the fields β, U and V) and one evolution equation (for γ), a wave equation. These equations are solved by the same marching algorithms described in GPW, now with additional source terms arising from the matter fields. As we are using spherical coordinates, we have to take special care with the numerical treatment of the coordinate singularities at the origin and the polar axis. To this aim we extend the treatment of the origin presented in GPW to nonvacuum spacetimes and regularize the poles by introducing a new azimuthal coordinate $y = -cos\theta$ and by redefining variables, for example $\gamma = \hat{\gamma}\ sin^2\theta$. As we want to keep the freedom of working with numerical grids which only cover the neutron star without its vacuum exterior, we generalize the radial coordinate used in GPW. Starting from an equidistant radial coordinate $x \in [0, 1]$, we allow for a general coordinate transformation of the form $r = r(x)$, which enables us to use compactified or non-compactified grids.

As already mentioned, we use HRSC schemes to solve the fluid equations (2) and (5). They constitute a hyperbolic system of balance laws. We use a so-called method of lines written in conservation form to solve these equations and to update the initial data forward in time. The principal part is solved by a Godunov-like method, with the same approximate Riemann solver as in reference[11] (see this reference for further details).

2 Tests and Applications

In this section, we apply our coupled code to models of neutron stars. For completeness we mention that the axisymmetric vacuum part of the code has been successfully tested in additional situations. These comprise the comparison to an exact solution (SIMPLE), the evolution of weak ingoing gravitational waves and an energy conservation test for vacuum data, following the tests described in [6].

2.1 Long-term stability of spherical neutron stars

As a simplified model for a self-gravitating neutron star we consider the spherically symmetric solution of the general relativistic hydrostatic equation with a polytropic equation of state, $p = K\rho^\Gamma$. This is the Tolman-Oppenheimer-Volkoff solution in its light cone representation [11]. In our analysis, we have worked with two representative models, following the choice of parameters made also in [3]. We restrict the discussion to the more relativistic one of these models. In dimensionless units ($c = G = M_\odot = 1$), the equilibrium properties of this star are described by: a polytropic index $\Gamma = 2$, a polytropic constant $K = 100$ and a central density $\rho_c = 1.28\ 10^{-3}$. The equilibrium model has a total mass $M = 1.4$. The time light needs to travel across this neutron star corresponds to about 17u in our time unit. We use this model for convergence tests as well as to check long-term stability of the numerical algorithms.

As the Tolman-Oppenheimer-Volkoff solution is static, convergence can be easily checked by subtracting the evolved solution from the initial (true) solution. We find second order convergence in all variables. Figure 2 shows a suitable norm of all variables as a function of the radial coordinate, which measures the deviation

from the initial solution for different grid resolutions.

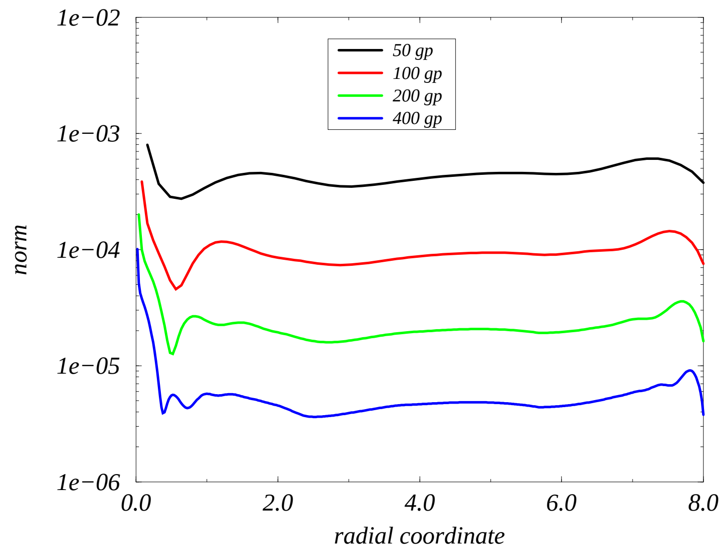

Figure 2. Convergence test for the Tolmann-Oppenheimer-Volkoff solution. Plotted is a suitable norm of all variables, which measures the deviation from equilibrium, for the indicated number of radial grid points (gp), after about two light-crossing times. To good approximation the solution is 2nd order convergent.

Figure 3 displays the L2-norm of the radial velocity $||u^r||_2$ of the star as a function of time. Due to discretization errors radial fluid motion is generated, the radial velocity slightly deviates from zero and the star oscillates in its radial modes of pulsation. These radial oscillations do not increase with time, which reflects the long-term stability of our numerical implementation.

Figure 4 shows the density profile of the neutron star for a very long integration time u=10000 for a lower resolution. Even though this corresponds to a very long-term, fully general relativistic hydrodynamic evolution, the density profile almost does not change, the equilibrium of the star being maintained to very good precision. The upper line corresponds to the initial model, the lowest line to the density profile at the time u=10000, which corresponds to roughly 590 light-crossing times.

2.2 Non-spherical matter motions

Next we consider the spherical model of the neutron star described in the previous section and add a non-spherical density component. This will lead to non-spherical evolution and hence allows for testing the non-spherical terms of the evolution system. More precisely, we add a Gaussian profile, which is centered at half radius at the polar axis and whose maximal value corresponds to 10% of the density of the

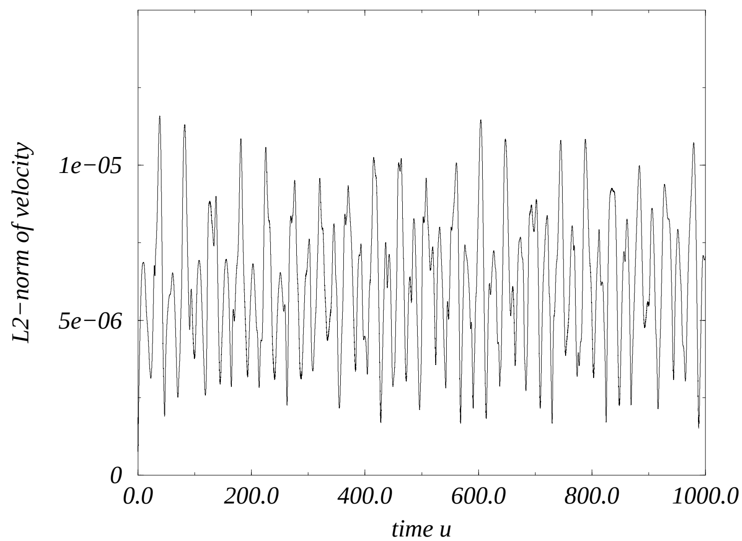

Figure 3. Radial velocity of the star, averaged over the radial coordinate, as a function of time. The evolution corresponds to roughly 59 light-crossing times.

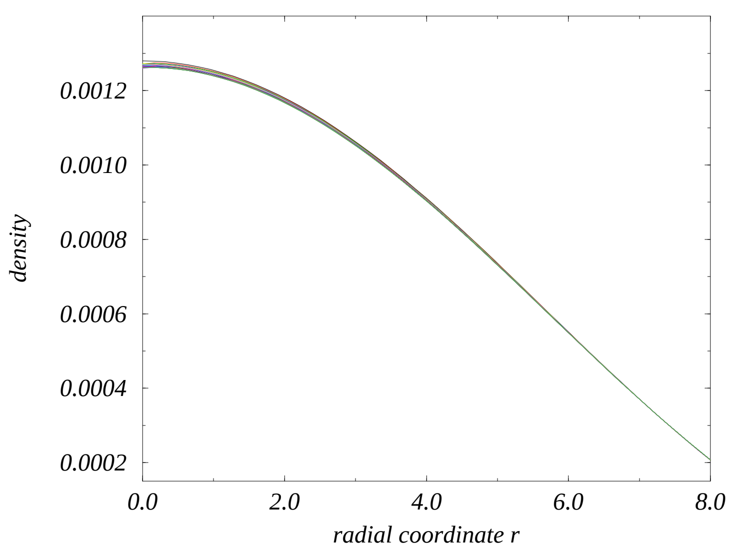

Figure 4. Density profile of the neutron star at the initial and subsequent times. The final time corresponds to $u = 10000$ (roughly 590 light-crossing times). The equilibrium model is maintained to a very good precision after such very long integration times.

equilibrium model at that point. Our initial model thus shows strong deviations from spherical symmetry. The non-spherical fluid motion will trigger gravitational waves. Figure 5 shows a series of snapshots of this evolution. We plot here the evolution of the overdensity, that is, the difference between the non-spherical solution and the equilibrium one. The evolution induces a large mass transfer. Due to this mass transfer, we would expect a remarkable gravitational wave signal, i.e. the quantity γ picks up a non-zero contribution. We found that our code is also second order convergent even for such non-spherical data. As we are in a regime where the exact solution is not known, we use Cauchy convergence for this test.

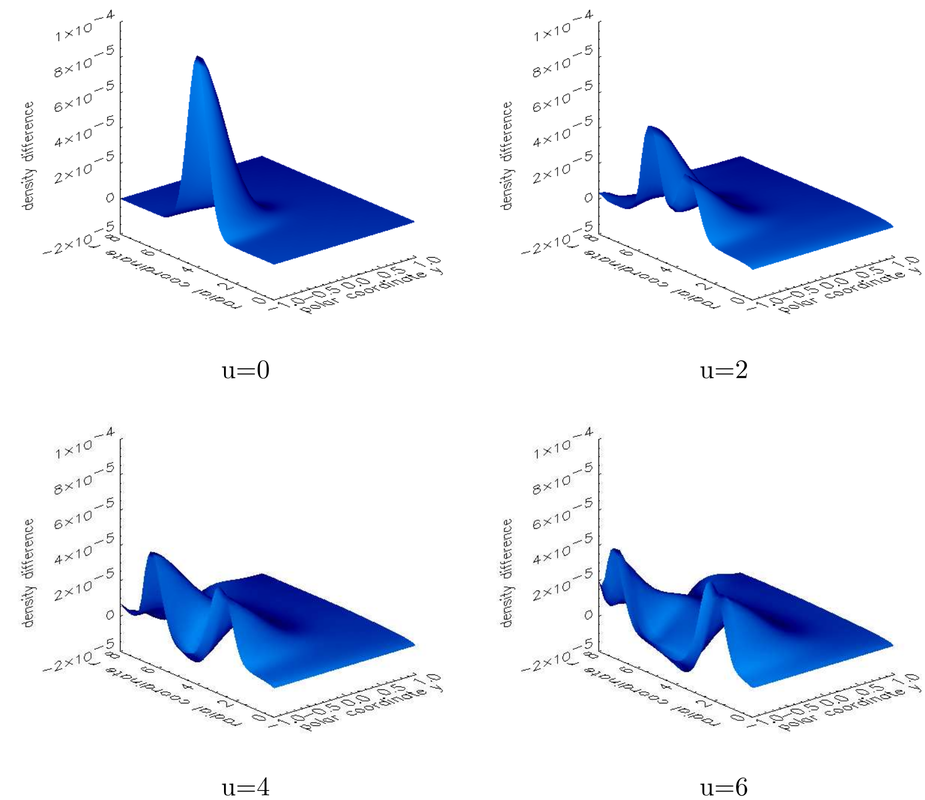

Figure 5. Evolution of a non-spherical density distribution on top of the equilibrium model of the neutron star. Plotted are the profile of the initial overdensity and the profiles of the overdensity at subsequent times as a function of the coordinates r and y. The evolution shows a large, non-spherical mass transfer, which allows us to check the second-order accurate implementation of all terms in our code.

2.3 Generation of gravitational waves

One of the major benefits of using lightlike foliations is the possibility of correctly extracting gravitational radiation after a compactification of spacetime. For an asymptotically flat spacetime, the metric fields can be Taylor expanded around future null infinity $\mathcal{I}^+$ (which corresponds to the limit $r \to \infty$ using our null foliation) in inverse powers of the radial coordinate r. Using the coefficients of this power series, the radiated energy emitted by gravitational waves can be determined.[8] In the same way, one could determine the Bondi mass, the total energy. However, as one would have to pick up nonleading terms in the power series expansion for the metric variable V, as the leading terms diverge on $\mathcal{I}^+$, the numerical extraction is easily spoiled by numerical errors. We therefore follow the work of Gómez et al[5] and globally introduce new metric variables, from which the Bondi mass can be determined using the leading term of their power series expansion.

In this section, we focus on a global energy conservation test. We start with a strongly perturbed neutron star and evolve this configuration for a very short time, which is sufficient for our purpose of testing convergence. We use a strong ingoing gravitational wave to perturb the equilibrium star,

$$\hat{\gamma} = 0.05 \; e^{-2(r-4)^2} \; e^{-4y^2}. \tag{7}$$

Such a large amplitude is not realistic, but we choose it to test our numerical implementation in the nonlinear regime. Such initial data involve large velocities in the fluid of the star and produce strong outgoing gravitational waves. By 'strong' we understand that the energy which is radiated away by gravitational waves is larger than the numerical errors for the calculation of the Bondi mass for the chosen resolution and integration time. Let M be the Bondi mass and P the total energy radiated away by gravitational waves. Thus, the convergence of the quantity

$$ec := M|_{u=0} - M|_{u=u_*>0} - P|_{[0,u_*]} \tag{8}$$

to zero represents a very severe global test for our implementation (see Figure 6).

The obtained first-order convergence rate can be explained by the use of a *total variation diminishing* HRSC scheme, which, although it is second-order accurate in smooth, monotonous parts of the flow, reduces to first-order at local extrema, which are present in the interior of the numerical domain in this test. Tests including propagation and scattering off the origin of pure vacuum gravitational fields yielded the expected second-order convergence.

There are two additional consistency conditions, which relate the metric quantities on $\mathcal{I}^+$.[10] These are

$$S = U_{,\theta} + U \; cot\theta \tag{9}$$

$$\gamma_{,u} = -\frac{1}{2} e^{-2\gamma} sin\theta (e^{2\gamma} \frac{U}{sin\theta})_{,\theta}, \tag{10}$$

where S is globally defined as

$$S = \frac{V-r}{r^2}. \tag{11}$$

Here again, we find first-order convergence.

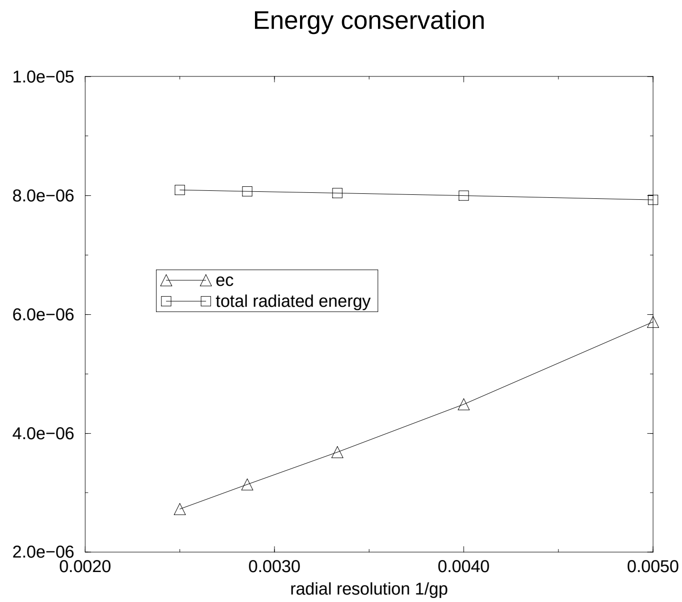

Figure 6. Global energy conservation test for a neutron star and a strong gravitational wave. Plotted are the deviation from energy conservation (triangles) and the total energy emitted by gravitational waves (squares). The final integration time is $u_* = 0.002$, the number of angular grid points is 0.2 gp + 1, where gp denotes the number of radial grid points.

3 Conclusion

We have developed an axisymmetric, fully general relativistic characteristic code with a perfect fluid matter field. We have checked convergence in spacetimes containing a neutron star. For a spherical, relativistic equilibrium model of a polytropic neutron star, we have shown long-term stability for hundreds of light-crossing times. In axisymmetry, we analyzed convergence in strongly perturbed neutron stars. Compactifying the spacetime, we could show global energy conservation, in the sense that the amount of energy lost by the system is radiated away by gravitational waves.

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